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1998 J. Phys. A: Math. Gen. 31 7791

(http://iopscience.iop.org/0305-4470/31/38/014)

J. Phys. A: Math. Gen. **31** (1998) 7791–7799. Printed in the UK PII: S0305-4470(98)90580-7

Application of path integration to operator calculus

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Received 12 January 1998

Abstract. We consider a disentanglement of the operator functions of the form $\gamma^\alpha \dots \gamma^\beta \exp\{\omega_{\mu\nu}\gamma^\mu\gamma^\nu\}$, where γ^μ are generating elements of a Clifford algebra (γ matrices, for example). To this end we formulate a path integral reduction procedure which allows one to obtain the functions under consideration in Sym-form. Then, by means of path integration, we obtain explicit decompositions of the operator functions in Sym-products of γ matrices (in the linearly independent γ-matrix structures) valid in arbitrary dimensions. Several particular examples are analysed in detail.

1. Introduction

As is well known, path integrals are widely and fruitfully applied in contemporary theoretical physics [1]. For example, they are used to solve the Schrödinger equation and the equations of diffusion theory, they are well adopted for quasiclassical calculations in quantum mechanics, they are used for the quantization of gauge theories and serve as the basic language in instanton physics, and they have found wide application in statistical mechanics, especially when methods of quantum field theory are used. The integrals over Grassmann variables introduced by Berezin [2] made it possible to define the corresponding path integrals over Grassmann-odd trajectories. This enlarged even more the field of application of path integrals [3]. In the present paper we would like to focus on the possibility of how one can use path integrals over Grassmann-odd trajectories to disentangle complicated functions on non-commuting operators (some rules of dealing with such functions were considered in [4–6]). Namely, we are going to consider the operator functions of the form

$$R_k = \underbrace{\gamma^\alpha \dots \gamma^\beta}_{k} \exp\{\omega_{\mu\nu}\gamma^\mu\gamma^\nu\} \qquad k < D \tag{1}$$

where the constant matrix ω is antisymmetric, $\omega_{\mu\nu} = -\omega_{\nu\mu}$, and γ^μ, $\mu = 0, 1, \dots, D-1$, are generating elements of some Clifford algebra,

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}. \tag{2}$$

The latter can be, in particular, understood as γ-matrices in D dimensions (in this case, $\eta_{\mu\nu} = \mathrm{diag}(1, -1, \dots, -1)$). Expressions of the form (1) frequently arise in different theoretical constructions. Here one ought to mention spinor representations of the Lorentz group. It is also known that propagators of relativistic spinning particles and superstrings in external fields, derived by means of the Schwinger proper-time method, contain γ matrices in the form (1). Performing calculations with propagators of that kind, one inevitably comes to the problem of the expansion of such expressions in terms of independent γ-matrix structures. One has also to mention that modern field theories and superstring theory are usually formulated in spacetime dimensions different from four. Thus, it is important to analyse the structure of the operator functions (1) for arbitrary dimensions. Moreover, commutation relations between the generators Γ_a of a Lie algebra can be realized by bilinear combinations of some Clifford algebra generating elements, similar to Schwinger-type representations via creation and annihilation operators [7]. Indeed, let, for example, Γ_a, $a = 1, \ldots, n$, be generators of $SU(N)$ group, $[\Gamma_a, \Gamma_b] = \mathrm{i} f_{abc}\Gamma_c$. Then one can see that the commutation relations of the algebra can be obeyed by means of the following representation: $\Gamma_a = -\frac{1}{4}\mathrm{i} f_{acd}\gamma_c\gamma_d$, where γ_a are generators of the corresponding Clifford algebra, $[\gamma_a, \gamma_b]_+ = \delta_{ab}$. Then finite transformations of the corresponding Lie group are presented by the operator functions R_0. Thus, the operator problem under consideration seems to be of current interest. We present a decomposition of the operator functions (1) via symmetrical (Sym) products of γ matrices which constitute linearly independent structures in a finite number. To do this we formulate a Grassmann path integral reduction procedure which allows one to obtain the functions under consideration in Sym-form. Then the problem can be solved by means of a path integration. Thus, we obtain the explicit γ-matrix structure of the operator functions under consideration in arbitrary dimensions. Finally, we consider particular cases in lower dimensions ($D = 3, 4$) identifying the corresponding decompositions with some known formulae derived by means of direct combinatoric methods strongly related to concrete properties of γ matrices in such dimensions. We find it remarkable that the solution of the operator problem is facilitated considerably by using the method of path integration. This extends the list of its useful applications.

2. *T* and Sym form of the operator functions

First, let us consider a particular case of the operator expression (1), namely, R_0. Using the famous Feynman consideration [5], we attach a continuous index t (we will call it time) to the operators and assume that the order in which the operators act is determined by the values of the indices ('the operator with higher time acts later') instead of the position of the operators on the paper. This chronological product will be indicated by $\mathcal{P}$, for example,

$$\mathcal{P}\sigma^{\mu\nu}(t_1)\sigma^{\kappa\lambda}(t_2) = \Theta(t_1 - t_2)\sigma^{\mu\nu}\sigma^{\kappa\lambda} + \Theta(t_2 - t_1)\sigma^{\kappa\lambda}\sigma^{\mu\nu}.$$

Under the sign of the chronological product the operators $\sigma^{\mu\nu}(t)$ commute and can be treated as ordinary c-numbers. With this in view and taking into account the fact that $\exp a \exp b = \exp(a + b)$ for a, b commuting, one obtains

$$R_0 = \mathcal{P}\ \exp\left\{\int_0^1 \omega_{\mu\nu}\sigma^{\mu\nu}(t)\,\mathrm{d}t\right\}. \tag{3}$$

One can note that expressions similar to (3) arise naturally in quantum-mechanical problems with Hamiltonians of the form $\mathcal{H}(t) = \mathrm{i}\omega_{\mu\nu}(t)\gamma^\mu\gamma^\nu$. In this case the evolution operator between the instants $t = 0$ and $t = 1$ has the form

$$U = \mathcal{P}\ \exp\left\{\int_0^1 \omega_{\mu\nu}(t)\sigma^{\mu\nu}(t)\,\mathrm{d}t\right\} \tag{4}$$

where the index t is now attached in a natural way to the σ matrices.

How do we calculate expression (3) efficiently? A convenient way is to use the Wick theorem [8] for appropriately defined T products of some operators whose commutators or anticommutators are c-numbers. In the case under consideration, γ matrices are such operators with anticommutators (2) being c-numbers. This dictates the choice of the 'fermionic' T product for γ matrices,

$$T\gamma^{\mu_1}(t_1)\dots\gamma^{\mu_n}(t_n) = \sum_P (-)^{\mathrm{sgn}(P)}\Theta(t_{P(1)},\dots,t_{P(n)})\gamma^{\mu_{P(1)}}\dots\gamma^{\mu_{P(n)}} \qquad n = 2, 3, \dots$$
$$T\gamma^\mu(\tau) = \gamma^\mu \qquad \Theta(t_1,\dots,t_n) = \Theta(t_1 - t_2)\dots\Theta(t_{n-1} - t_n) \tag{5}$$

where $\mathrm{sgn}(P)$ stays for the parity of the permutation P. In the T product the $\gamma^\mu(t)$ anticommute, i.e. they behave like Grassmann-odd objects. Another product of γ matrices in which they have the same behaviour is the symmetrical product,

$$\mathrm{Sym}\,\gamma^{\mu_1}\dots\gamma^{\mu_n} = \frac{1}{n!}\sum_P (-)^{\mathrm{sgn}(P)}\gamma^{\mu_{P(1)}}\dots\gamma^{\mu_{P(n)}} \qquad n = 1, 2, \dots$$
$$\mathrm{Sym}\,\gamma^\mu = \gamma^\mu. \tag{6}$$

In contrast with the case of the T product, γ matrices in the Sym products carry discrete indices only and the latter take a finite number D of values. Hence, due to the antisymmetry of (6) under permutations of the indices, every Sym product of more than D γ matrices vanishes. The unique (up to permutations) non-vanishing Sym product of γ matrices, $\mathrm{Sym}\,\gamma^0\dots\gamma^{D-1}$, in the case of D odd coincides with the identity operator 1 due to the anticommutation relations (2). For D even, the matrix $\gamma^D = \gamma^0\dots\gamma^{D-1}$ is distinct from 1. So, in any dimension D the identity 1 and the matrices

$$\mathrm{Sym}\,\gamma^{\mu_1}\dots\gamma^{\mu_k} \qquad \mu_1 < \mu_2 < \dots < \mu_k, k = 1, 2, \dots, 2\left[\frac{D}{2}\right]$$

form a basis in the associative algebra generated by $\gamma^0,\dots,\gamma^{D-1}$ and will be referred to as independent γ-matrix structures [9]. A modification of the Wick theorem allows one to express the T products in terms of Sym products of γ matrices. The difference between the T product and the Sym product of two γ matrices (the contraction), being proportional to their anticommutator, is a c-number,

$$T\gamma^{\mu_1}(t_1)\gamma^{\mu_1}(t_2) = \mathrm{Sym}\,\gamma^{\mu_1}\gamma^{\mu_2} + \Delta^{\mu_1\mu_2}(t_1,t_2)$$
$$\Delta^{\mu_1\mu_2}(t_1,t_2) = \eta^{\mu_1\mu_2}\epsilon(t_1 - t_2) \qquad \epsilon(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0. \end{cases} \tag{7}$$

Let a functional

$$F[\zeta] = \sum_n \int_0^1 \mathrm{d}t_1 \dots \int_0^1 \mathrm{d}t_n\, f_{\mu_1\dots\mu_n}(t_1\dots t_n)\zeta^{\mu_1}(t_1)\dots\zeta^{\mu_n}(t_n) \tag{8}$$

on the space of Grassmann-odd valued functions $\xi^\mu(t)$ be given. Then the matrix $TF[\gamma]$ can be presented as a series in Sym products

$$TF[\gamma] = \mathrm{Sym}\left[\exp\left\{-\frac{1}{2}\frac{\delta_\ell}{\delta\zeta_\mu}\star\Delta^{\mu\nu}\star\frac{\delta_\ell}{\delta\zeta_\nu}\right\}F[\zeta]\Big|_{\zeta(t)=\gamma}\right] \tag{9}$$

where $\delta_\ell/\delta\zeta^\mu$ stays for the left derivative, and a condensed notation is used in which the integrations over time are denoted by a star, i.e.

$$\frac{\delta_\ell}{\delta\zeta^\mu}\star\Delta^{\mu\nu}\star\frac{\delta_\ell}{\delta\zeta^\nu} = \int_0^1 \mathrm{d}t_1\int_0^1 \mathrm{d}t_2\frac{\delta_\ell}{\delta\zeta^\mu(t_1)}\Delta^{\mu\nu}(t_1,t_2)\frac{\delta_\ell}{\delta\zeta^\nu(t_2)}.$$

Sometimes discrete indices will also be omitted. In this case all tensors of second rank have to be understood as matrices with lines marked by the first contravariant indices of the tensors and with columns marked by the second covariant indices of the tensors.

The representation (9) is a functional formulation of the Wick theorem (Hori procedure [10]), modified to the fermionic case and to the transition from T to Sym product [11]. To use the Wick theorem (9) in the problem at hand we may replace the $\mathcal{P}$ product in (3) for the T product,

$$\mathcal{P}\ \exp\left\{\int_0^1 \omega_{\mu\nu}\sigma^{\mu\nu}(t)\,\mathrm{d}t\right\} = T\ \exp\left\{\int_0^1 \omega_{\mu\nu}\gamma^{\mu}(t)\gamma^{\nu}(t)\,\mathrm{d}t\right\}. \tag{10}$$

To justify the formula (10) one also has to define the T product for coinciding values of some continuous indices (the chronological prescription (6) fails to do it) and then to check (10) itself. It is convenient to define the T product for all time values by

$$T\,\gamma^{\mu_1}(t_1)\ldots\gamma^{\mu_n}(t_n) = \mathrm{Sym}\left[\exp\left\{-\frac{1}{2}\frac{\delta_\ell}{\delta\zeta^{\mu}}\star\Delta^{\mu\nu}\star\frac{\delta_\ell}{\delta\zeta^{\nu}}\right\}\zeta^{\mu_1}(t_1)\ldots\zeta^{\mu_n}(t_n)\bigg|_{\zeta=\gamma}\right] \quad n=1,2,\ldots \tag{11}$$

where $\Delta^{\mu\nu}$ is given by (7), $\Delta^{\mu\nu}(t,t) = \eta^{\mu\nu}\epsilon(0)$ and some finite value has been assigned to $\epsilon(0)$. Due to the Wick theorem (9), this definition is compatible with the chronological prescription (5). Using (11) one obtains

$$T\gamma^{\mu_1}(t_1)\gamma^{\nu_1}(t_1)\ldots\gamma^{\mu_n}(t_n)\gamma^{\nu_n}(t_n) = \mathcal{P}(\sigma^{\mu_1\nu_1}(t_1)+\epsilon(0))\ldots(\sigma^{\mu_n\nu_n}(t_n)+\epsilon(0)) \tag{12}$$

where the times $t_1,\ldots,t_n$ are supposed to be distinct. Substituting (12) into $T\exp\{\omega_{\mu\nu}\gamma^{\mu}\star\gamma^{\nu}\}$ one finds that the terms depending on $\epsilon(0)$ vanish due to the antisymmetry of ω, and equation (10) takes place independently of the value assigned to $\epsilon(0)$.

3. Path integral formulation of the Hori procedure

Wick theorem (9) admits a path-integral formulation. We define Gaussian and quasi-Gaussian path integrals over a space of Grassmann-odd trajectories in the framework of the perturbation theory approach [12–14]. The first one is defined as

$$\begin{aligned} I(K,\rho,E) &= \int_E D\xi\ \exp\{\tfrac{1}{4}\xi^{\mu}\star K_{\mu\nu}\star\xi^{\nu}+\rho_\mu\star\xi^{\mu}\} \\ &= \Lambda\,\mathrm{Det}\,K^{1/2}\exp\{\rho_\mu\star G^{\mu\nu}\star\rho_\nu\} \end{aligned} \tag{13}$$

where $\xi^{\mu}(t)$ are Grassmann-odd trajectories of integration, $\rho_\mu(t)$ are Grassmann-odd sources, K is a Grassmann-even antisymmetric kernel $K_{\mu\nu}(t,t') = -K_{\nu\mu}(t',t)$, $G^{\mu\nu}(t,t')$ is an inverse kernel (Green function),

$$\int_0^1 \mathrm{d}t'\,K_{\mu\nu}(t,t')G^{\nu\lambda}(t',t'') = \delta_\mu^\lambda\delta(t'-t'') \tag{14}$$

and Λ is a numerical factor which contains no parameters essential to the theory (parameters defining the matrices $K_{\mu\nu}(t,t')$). In general, equation (14) has more than one solution and $G(t,t')$ is specified by imposing some boundary conditions. In a natural way these boundary conditions can be understood as defining the space of integration E. In particular, the kernel K is not degenerate on E, i.e. the homogeneous equation $\int_0^1 \mathrm{d}t'\,K_{\mu\nu}(t,t')\xi^{\nu}(t') = 0$ does not have non-trivial solutions in E. Thus, equation (14) for the Green function has a unique solution. One can understand the space E as a function of the form $\xi^{\mu}(t) = \int_0^1 \mathrm{d}t'\,K_{\mu\nu}(t,t')\rho^{\nu}(t')$, where ρ belongs to the space of sources [11]. In this case the invariance of the space E under the shifts on such functions is a trivial fact which is important for efficient manipulations with the integrals under consideration. The quasi-Gaussian path integrals are defined via the Gaussian ones by the prescription

$$\int_E D\xi \; \exp\left\{\frac{1}{4}\xi^\mu \star K_{\mu\nu} \star \xi^\nu + \rho_\mu \star \xi^\mu\right\} F[\xi] = F\left[\frac{\delta_l}{\delta\rho}\right] I(K, \rho, E) \tag{15}$$

where $F[\xi]$ are arbitrary analytic functionals on E and $\delta_l/\delta\rho$ stand for the left derivatives. In the construction under consideration we encounter matrices $K_{\mu\nu}(t, t')$, part of the indices of which are continuous. To avoid problems with the calculation of the determinants of such matrices, as well as problems with the definition of the factor Λ, we may consider the relative quantities

$$\frac{I(K, \rho, E)}{I(K_0, 0, E)} = \mathrm{Det}(K/K_0)^{\frac{1}{2}} \; \exp\{\rho_\mu \star G^{\mu\nu} \star \rho_\nu\} \tag{16}$$

which are sufficient for our purposes. The matrix K_0 can often be chosen in a form so as to simplify the calculation of the determinant $\mathrm{Det}(K/K_0)$ (see later).

We will use two properties of the quasi-Gaussian path integrals which can be checked using the given definitions. First, the Gaussian path integral can be expressed as a quasi-Gaussian one,

$$I(K, \rho, E) = \exp\left\{\frac{1}{4}\frac{\delta_\ell}{\delta\rho_\mu} \star (K - K_0)^{\mu\nu} \star \frac{\delta_\ell}{\delta\rho_\nu}\right\} \; I(K_0, \rho, E) \tag{17}$$

provided both Gaussian integrals $I(K, \rho, E)$ and $I(K_0, \rho, E)$ exist. Second, quasi-Gaussian path integrals are invariant under the shifts, i.e.

$$\int_E D\xi \; \exp\{\tfrac{1}{4}(\xi + \zeta)^\mu \star K_{\mu\nu} \star (\xi + \zeta)^\nu\} F[\xi + \zeta] = \int_E D\xi \; \exp\{\tfrac{1}{4}\xi^\mu \star K_{\mu\nu} \star \xi^\nu\} \, F[\xi] \tag{18}$$

where ζ^μ is an arbitrary trajectory from E.

The path-integral formulation of the Wick theorem (9) is based on the following representation of the quadratic exponent,

$$\exp\{\rho_\mu \star G^{\mu\nu} \star \rho_\nu\} = \frac{I(K, \rho, E)}{I(K, 0, E)}. \tag{19}$$

Choosing $G^{\mu\nu}(t, t') = -\frac{1}{2}\Delta^{\mu\nu}(t, t')$ (where Δ is given by (7)), the matrix K is easily recognized to be $(K_0)_{\mu\nu}(t, t') = -\eta_{\mu\nu}\delta'(t - t')$, and the space E is determined by the boundary condition satisfied by Δ,

$$\Delta^{\mu\nu}(0, t) + \Delta^{\mu\nu}(1, t) = 0 \qquad 0 < t < 1. \tag{20}$$

According to the definition given, E in (19) is the space of Grassmann-odd trajectories $\xi^\mu(t)$ obeying the antiperiodic boundary condition

$$\xi(0) + \xi(1) = 0. \tag{21}$$

Replacing the odd sources $\rho_\mu(t)$ in (19) by left derivatives and applying the operator obtained to a functional $F[\zeta]$, one gets

$$\exp\left\{-\frac{1}{2}\frac{\delta_\ell}{\delta\zeta_\mu} \star \Delta^{\mu\nu} \star \frac{\delta_\ell}{\delta\zeta_\nu}\right\} F[\zeta] = \int_{\xi(0)+\xi(1)=0} \mathcal{D}\xi \; \exp\left\{-\frac{1}{4}\xi \star \dot{\xi}\right\} F[\xi + \zeta] \tag{22}$$

where

$$\mathcal{D}\xi = \frac{D\xi}{\int_{\xi(0)+\xi(1)=0} D\xi \; \exp\{-\frac{1}{4}\xi \star \dot{\xi}\}}. \tag{23}$$

Using equation (22) one can present the Wick theorem (9) in the form

$$TF[\gamma] = \mathrm{Sym}\left[\int_{\xi(0)+\xi(1)=0} \mathcal{D}\xi \; \exp\{-\tfrac{1}{4}\xi \star \dot{\xi}\} F[\xi + \zeta]|_{\zeta=\gamma}\right]. \tag{24}$$

4. Reduction of the operator functions

Choosing the functional $F[\zeta]$ in (24) of the form

$$F[\zeta] = \exp\left\{\int_0^1 \omega_{\mu\nu}\zeta^\mu(t)\zeta^\nu(t)\,\mathrm{d}t\right\}$$

and using (10), one gets the following representation for the matrix R_0

$$R_0 = \mathrm{Sym}\left[\int_{\xi(0)+\xi(1)=0} \mathcal{D}\xi\ \exp\{-\tfrac{1}{4}\xi \star \dot{\xi}\}\exp\{\omega_{\mu\nu}(\xi+\zeta)^\mu \star (\xi+\zeta)^\nu\}|_{\zeta=\gamma}\right]. \tag{25}$$

The quasi-Gaussian path integral in (25) can be understood as a Gaussian one due to the property (17). Taking into account equation (23), one obtains

$$R_0 = \mathrm{Sym}\left[\frac{I(K_\omega, 2\zeta\omega, E)}{I(K_0, 0, E)}\exp(\omega_{\mu\nu}\zeta^\mu \star \zeta^\nu)|_{\zeta=\gamma}\right] \tag{26}$$

where

$$K_\omega(t,t') = -\eta\delta'(t-t') + 4\omega\delta(t-t'). \tag{27}$$

Evaluating the ratio of the path integrals in (26) by means of (16) and setting $\zeta^\mu(t) = \gamma^\mu$ one obtains

$$R_0 = \left(\mathrm{Det}\frac{K_\omega}{K_0}\right)^{1/2} \mathrm{Sym}\ \exp\{M_{\mu\nu}\gamma^\mu\gamma^\nu\} \tag{28}$$

where

$$M_{\mu\nu} = \omega_{\mu\nu} - 4\omega_{\mu\kappa} \star G_\omega^{\kappa\lambda} \star \omega_{\lambda\nu} \tag{29}$$

G_ω being the Green function for K_ω,

$$\int_0^1 (K_\omega)_{\mu\nu}(t,t')G_\omega^{\nu\lambda}(t',t'') = \delta_\mu^{\nu\lambda}(t,t'')$$

which obeys the boundary condition (20). Evaluating

$$G_\omega(t,t') = -\tfrac{1}{2}\,\mathrm{e}^{4\omega(t-t')}(\epsilon(t-t') - \tanh 2\omega)$$

and substituting in (28) we find

$$M = \tfrac{1}{2}\tanh 2\omega. \tag{30}$$

Calculating the determinant

$$\mathrm{Det}(K_\omega K_0^{-1}) = \exp\mathrm{Tr}\left\{4\omega\int_0^1 G_{s\omega}\,\mathrm{d}s\right\} = \det\cosh 2\omega \tag{31}$$

and substituting (30) and (31) into (28) we finally get

$$\begin{aligned} R_0 &= \exp\{\omega_{\mu\nu}\gamma^\mu\gamma^\nu\} \\ &= (\det\cosh 2\omega)^{1/2}\mathrm{Sym}\ \exp\{\tfrac{1}{2}(\tanh 2\omega)_{\mu\nu}\gamma^\mu\gamma^\nu\}. \end{aligned} \tag{32}$$

A remarkable feature of the expansion on the right-hand side of equation (32) is that it contains only a finite number of terms. Indeed, every Sym product of more than D γ matrices vanishes. We have found, in fact, an explicit decomposition, valid in any dimension, of the spinor representation matrix $\exp\{\omega_{\mu\nu}\gamma^\mu\gamma^\nu\}$ for the Lorentz transformation $L = \exp 4\omega$ in terms of the independent γ-matrix structures.

Taking $D = 3$ where, for example, $\gamma^0 = \sigma^3$, $\gamma^1 = \mathrm{i}\sigma^1$, $\gamma^2 = \mathrm{i}\sigma^2$, we get

$$R_0 = \exp\{\omega_{\mu\nu}\gamma^\mu\gamma^\nu\} = (\det\cosh 2\omega)^{1/2}[1 + \tfrac{1}{2}(\tanh 2\omega)_{\mu\nu}\gamma^\mu\gamma^\nu] \tag{33}$$

which can be easily transformed to the familiar form

$$\exp\left\{\frac{\mathrm{i}}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right\} = \cos\frac{\theta}{2} + \mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\frac{\theta}{2} \qquad \boldsymbol{\theta} = \theta\boldsymbol{n} \qquad \boldsymbol{n}^2 = 1$$

where

$$\theta^2 = \sum_{i=1}^{3}\theta_i^2 \qquad \theta_1 = 4\mathrm{i}\omega_{20} \qquad \theta_2 = 4\mathrm{i}\omega_{01} \qquad \theta_3 = -4\omega_{12}.$$

In the case $D = 4$ one obtains

$$\begin{aligned} R_0 &= \exp\{\omega_{\mu\nu}\gamma^\mu\gamma^\nu\} \\ &= (\det\cosh 2\omega)^{1/2}[1 + \tfrac{1}{2}(\tanh 2\omega)_{\mu\nu}\sigma^{\mu\nu} + \tfrac{1}{8}\epsilon^{\kappa\lambda\mu\nu}(\tanh 2\omega)_{\kappa\lambda}(\tanh 2\omega)_{\mu\nu}\gamma^5] \end{aligned} \tag{34}$$

where $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ and $\epsilon^{\kappa\lambda\mu\nu}$ is the Levi-Civita symbol normalized by $\epsilon^{0123} = 1$. A different form of the decomposition on the left-hand side of (34) was obtained in [15] using a direct combinatoric method and concrete properties of γ matrices in four dimensions,

$$\begin{aligned} R_0 &= [16G(L)]^{-1/2}[G(L) + \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}L^{\mu\nu}L^{\rho\sigma}\gamma^5 - (L^2)_{\mu\nu}\sigma^{\mu\nu} + (2 + \mathrm{tr}\, L)L_{\mu\nu}\sigma^{\mu\nu}] \\ G(L) &= 2(1 + \mathrm{tr}\, L) + \tfrac{1}{2}(\mathrm{tr}\, L)^2 - \tfrac{1}{2}\mathrm{tr}\, L^2. \end{aligned} \tag{35}$$

The equivalence of the decompositions (34) and (35) can be checked by a straightforward, although long, calculation which we do not present here. We stress again that the derivation in paper [15] is strongly related to $D = 4$ and its generalization to other dimensions is not clear.

To disentangle more complicated operator functions, in particular those of the form (1), it is convenient to introduce the generating functional

$$J[\rho,\zeta] = \int_{\xi(0)+\xi(1)=0}\mathcal{D}\xi\ \exp\{-\tfrac{1}{4}\xi\star\dot{\xi} + \omega_{\mu\nu}(\xi+\zeta)^\mu\star(\xi+\zeta)^\nu + \rho_\mu\star(\xi+\zeta)^\mu\}. \tag{36}$$

Then

$$R_k = \lim_{t_k\to 1}\ldots\lim_{t_1\to 1}\mathrm{Sym}\left[\frac{\delta_\ell^k}{\delta\rho_\alpha(t_1)\ldots\delta\rho_\beta(t_k)}J[\rho,\zeta]|_{\rho=0;\zeta=\gamma}\right]. \tag{37}$$

Taking into account (23), the generating functional $J[\rho,\zeta]$ is calculated by means of (16) and (31) to be

$$J[\rho,\zeta] = (\det\cosh 2\omega)^{1/2}\exp\{(\rho + 2\zeta\omega)_\mu\star G_\omega^{\mu\nu}\star(\rho - 2\omega\zeta)_\nu + \omega_{\mu\nu}\zeta^\mu\star\zeta^\nu\}. \tag{38}$$

Using equations (37) and (38) one finds a formula which is valid in any dimension

$$R_1 = \gamma^\alpha\exp\{\omega_{\mu\nu}\gamma^\mu\gamma^\nu\} = \mathrm{Sym}[(\eta + \tanh 2\omega)^{\alpha\kappa}\gamma_\kappa\exp\{\tfrac{1}{2}(\tanh 2\omega)_{\mu\nu}\gamma^\mu\gamma^\nu\}]. \tag{39}$$

For $D = 4$ the expression on the right-hand side reduces to

$$\begin{aligned} R_1 &= \gamma^\alpha\exp\{\omega_{\mu\nu}\gamma^\mu\gamma^\nu\} \\ &= (\eta + \tanh 2\omega)^{\alpha\kappa}\gamma_\kappa + \tfrac{1}{2}\epsilon^{\kappa\mu\nu\lambda}(\eta + \tanh 2\omega)^\alpha{}_\kappa(\tanh 2\omega)_{\mu\nu}\gamma^5\gamma_\lambda. \end{aligned} \tag{40}$$

Another representation for the left-hand side of (40) has been derived in $D = 4$ using concrete properties of γ matrices in such dimensions [16],

$$R_1 = \gamma^\alpha\exp\{\omega_{\mu\nu}\gamma^\mu\gamma^\nu\} = (\mathrm{e}^{2\omega}\cos 2\omega^*)^\alpha{}_\kappa\gamma^\kappa + (\mathrm{e}^{2\omega}\sin 2\omega^*)^\alpha{}_\kappa\gamma^5\gamma^\kappa. \tag{41}$$

One can prove the equivalence of both decompositions (40) and (41).

As was mentioned in the introduction, operator expressions of the form (1) often appear in different constructions, especially in quantum field theory. Their decompositions in independent γ-matrix structures are necessary for concrete calculations. A simple example gives us the Dirac propagator of a spinning particle in a constant uniform electromagnetic field, which was calculated first by Schwinger [17] in four dimensions:

$$S_0^c(x_{\text{out}}, x_{\text{in}}) = \left[\gamma^\mu\left(\mathrm{i}\frac{\partial}{\partial x^\mu_{\text{out}}} - eA_\mu(x_{\text{out}})\right) + m\right]\int_0^\infty \mathrm{d}s\, g(x_{\text{out}},\, x_{\text{in}},\, s) \tag{42}$$

where the transformation function g has the form

$$g(x_{\text{out}},\, x_{\text{in}},\, s) = \frac{1}{16\pi^2}\left(\det\frac{\sinh eFs}{eF}\right)^{-1/2}\exp\left\{\mathrm{i}\frac{e}{2}x_{\text{out}}Fx_{\text{in}} - \mathrm{i}sm^2 - \mathrm{i}\frac{e}{4}(x_{\text{out}} - x_{\text{in}})F\right.$$
$$\left.\times\coth(eFs)(x_{\text{out}} - x_{\text{in}}) + \frac{es}{2}F_{\mu\nu}\sigma^{\mu\nu}\right\} \tag{43}$$

and contains an operator construction of the form R_0. By means of the formula (34) one can obtain the explicit γ-matrix structure of the transformation function to be

$$g(x_{\text{out}},\, x_{\text{in}},\, s) = \frac{1}{16\pi^2}\left(\det\frac{\tanh eFs}{eF}\right)^{-1/2}$$
$$\times\exp\left\{\mathrm{i}\frac{e}{2}x_{\text{out}}Fx_{\text{in}} - \mathrm{i}sm^2 - \mathrm{i}\frac{e}{4}(x_{\text{out}} - x_{\text{in}})F\coth(eFs)(x_{\text{out}} - x_{\text{in}})\right\}$$
$$\times\left[1 + \frac{1}{2}(\tanh eFs)_{\mu\nu}\sigma^{\mu\nu} + \frac{1}{8}\epsilon^{\alpha\beta\mu\nu}(\tanh eFs)_{\alpha\beta}(\tanh eFs)_{\mu\nu}\gamma^5\right]. \tag{44}$$

Acknowledgments

DMG and PBB thank the Brazilian Foundation CNPq for support. SIZ thanks the Departamento de Física Matemática at Universidade de São Paulo for hospitality.

References

[1] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
Arthurs A M (ed) 1975 *Functional Integration and its Applications* (Oxford: Clarendon)
Papadopoulis G J and Devreese J T (eds) 1978 *Path Integrals and their Applications in Quantum, Statistical and Solid State Physics (Proc. NATO Adv. Study Ins., Antwerp, 1977)* (New York: Plenum)
Berezin F A and Shubin M A 1983 *The Schrödinger Equation* (Moscow: Moscow State University Press)
Simon B 1979 *Functional Integration in Quantum Physics* (New York: Academic)
DeWitt-Morette C, Maheshwari A and Nelson B 1979 Path integration in non-relativistic quantum mechanics *Phys. Rep.* **50C** 255
Marinov M S 1980 Path integrals in quantum theory: an outlook of basic concepts *Phys. Rep.* **60** 1–57
Berezin F A 1980 Feynman path integrals in a phase space *Uspekhi Fiz. Nauk* **132** 497 (Engl. Trans. 1991 *Sov. Phys. Usp.* **23** 763)
Schulman L S 1981 *Techniques and Applications of Path Integration* (New York: Wiley–Interscience)
Rivers R J 1987 *Path Integral Methods in Quantum Field Theory* (Cambridge: Cambridge University Press) pp 1–339

[2] Berezin F A 1965 *The Method of Second Quantization* (Moscow: Nauka) (1966 New York: Academic)
Berezin F A 1987 *Introduction to Superanalysis* (Dordrecht: Reidel)
DeWitt B 1985 *Supermanifolds* (Cambridge: Cambridge University Press)

[3] Berezin F A and Marinov M S 1975 *Pisma Zh. Eksp. Theor. Fiz.* **21** 678 (Engl. Transl. 1975 *JETP Lett.* **21** 320)
Berezin F A and Marinov M S 1977 *Ann. Phys., NY* **104** 336
Ogielski A T and Sobczuk J 1981 *J. Math. Phys.* **22** 2060
Henneaux M and Teitelboim C 1982 *Ann. Phys.* **143** 127
Borisov N V and Kulish P P 1982 *Teor. Math. Fiz.* **51** 335

Polyakov A M 1987 *Gauge Fields and Strings* (Chur, Switzerland: Harwood)
Fainberg V Ya and Marshakov A V 1988 *JETP Lett.* **47** 565
Fainberg V Ya and Marshakov A V 1988 *Phys. Lett.* **211B** 81
Fainberg V Ya and Marshakov A V 1988 *Nucl. Phys.* B **306** 659
Fainberg V Ya and Marshakov A V 1990 *Proc. PhIAN* **201** 139 (Moscow: Nauka)
Aliev T M, Fainberg V Ya and Pak N K 1994 *Nucl. Phys.* B **429** 321
Fradkin E S and Gitman D M 1991 *Phys. Rev.* D **44** 3230
van Holten J W 1995 *Nucl. Phys.* B **457** 375
van Holten J W NIKHEF-H/95-055
van Holten J W 1996 *Proc. 29th Ahrenshoop Symp. on the Theory of Elementary Particles, Buckow, Germany 1995* ed D Lust and G Weigt (Amsterdam: North-Holland)
Gitman D 1997 *Nucl. Phys.* B **488** 490
[4] Bloch F 1932 *Z. Phys.* **74** 295
[5] Feynman R P 1951 *Phys. Rev.* **84** 108
[6] Kirznitz D A 1951 *Proc. P.N. Lebedev Phys. Inst.* **XVI** 3
Kirznitz D A 1963 *Field Methods in the Theory of Many Particles* (Moscow: Atomizdat)
[7] Schwinger J 1965 *Quantum Theory of Angular Momentum* ed L C Biedenharn and H van Dam (New York: Academic)
[8] Wick D 1950 *Phys. Rev.* **80** 268
[9] Brauer R and Weyl H 1935 *Am. J. Math.* **57** 425
[10] Hori S 1952 *Prog. Theor. Phys.* **7** 578
[11] Vasiliev A N 1976 *Functional Methods in Quantum Field Theory and Statistics* (Leningrad: Leningrad State University Press)
[12] Slavnov A A 1975 *Teor. Mat. Fiz.* **22** 177
[13] Slavnov A A and Faddeeev L D 1978 *Introduction into Quantum Theory of Gauge Fields* (Moscow: Nauka)
[14] Gitman D M and Tyutin I 1991 *Quantization of Fields with Constraints* (Berlin: Springer)
[15] Macfarlane A J 1966 *Commun. Math. Phys.* **2** 133
[16] Baier V N, Katkov V M and Strakhovenko V M 1975 *Sov. Phys.–JETP* **41** 198
[17] Schwinger J 1951 *Phys. Rev.* **82** 664